# Monoids and Modules form a Cartesian Equipment

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**Abstract**

This is a work-in-progress meant to give a complete proof of the fact that monoids, homomorphisms, modules and so-called modulations in a suitably structured double category form a cartesian equipment. It bears repeating that this is an evolving document.

## Contents



## 1 Background, Motivation, and Overview

## 2 Preliminary Definitions

We shall start from the given notion of a *double category* – namely, a pseudo-category in the 2-category of categories, functors and natural transformations. Generic double categories are thus always assumed to be *pseudo* and if they are supposed to be strict, we shall say so. As a matter of notation, we shall write cells of a given generic double category in the form

$$\begin{array}{ccc} X & \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & Y \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ Z & \underset{N}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & W \end{array}$$

Here $M$ and $N$ are *proarrows* giving the internal domain and codomain of the cell $\alpha$, $f$ and $g$ are ordinary arrows, the external *source* and *target* of the cell $\alpha$. In the interest of economy, in large diagrams we shall sometimes leave out the objects as for example in

$$\begin{array}{ccc} \cdot & \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \cdot \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ \cdot & \underset{N}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \cdot \end{array}$$

especially if these objects are understood or easily recovered from the labelled arrows and proarrows. When they are needed, we shall use small gothic letters such as '$\mathfrak{a}$', '$\mathfrak{l}$', and '$\mathfrak{r}$' for associators and unitors coming with the pseudo structure associated to a given double category. General references for the rest of the material in this section include the papers of Grandis & Paré [GP99], [GP04] and well as the textbook of Grandis [Gra20]. We will cite other specialized sources in discussing further definitions and out-of-the-way examples.

**Definition 2.1.** A **lax functor** $F\colon \mathbb{X} \to \mathbb{D}$ consists of two functors $F_0\colon \mathbb{X}_0 \to \mathbb{D}_0$ and $F_1\colon \mathbb{X}_1 \to \mathbb{D}_1$ as well as canonical comparison cells

$$
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}} & FX \\
\| & F_X & \| \\
FX & \xrightarrow[F(y_X)]{} & FX
\end{array}
\qquad\qquad
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ \\
\| & & F_{M,N} & & \| \\
FX & & \xrightarrow[F(M\otimes N)]{} & & FZ
\end{array}
$$

satisfying the following conditions.

1. $F_0$ and $F_1$ are compatible with external sources and targets in the sense that both

$$
\begin{array}{ccc}
\mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{D}_1 \\
{\scriptstyle \mathrm{src}}\downarrow & & \downarrow{\scriptstyle \mathrm{src}} \\
\mathbb{X}_0 & \xrightarrow[F_0]{} & \mathbb{D}_0
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{D}_1 \\
{\scriptstyle \mathrm{tgt}}\downarrow & & \downarrow{\scriptstyle \mathrm{tgt}} \\
\mathbb{X}_0 & \xrightarrow[F_0]{} & \mathbb{D}_0
\end{array}
$$

   commute.

2. The laxity comparison cells are natural in the sense that for any ordinary arrow $f$

$$
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}} & FX \\
\| & F_X & \| \\
FX & \xrightarrow{Fy_X} & FX \\
{\scriptstyle Ff}\downarrow & Fy_f & \downarrow{\scriptstyle Ff} \\
FY & \xrightarrow[Fy_Y]{} & FY
\end{array}
\quad = \quad
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}} & FX \\
{\scriptstyle Ff}\downarrow & y_{Ff} & \downarrow{\scriptstyle Ff} \\
FY & \xrightarrow{y_{FY}} & FY \\
\| & F_Y & \| \\
FY & \xrightarrow[Fy_Y]{} & FY
\end{array}
$$

   and for any cells $\alpha$ and $\beta$

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ \\
\| & & F_{M,N} & & \| \\
FX & & \xrightarrow[F(M\otimes N)]{} & & FZ \\
{\scriptstyle Ff}\downarrow & & F(\alpha\otimes\beta) & & \downarrow{\scriptstyle Fh} \\
FX' & & \xrightarrow[F(P\otimes Q)]{} & & FZ'
\end{array}
\quad = \quad
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ \\
{\scriptstyle Ff}\downarrow & F\alpha & \downarrow{\scriptstyle Fg} & F\beta & \downarrow{\scriptstyle Fh} \\
FX' & \xrightarrow[FP]{} & FY' & \xrightarrow[FQ]{} & FZ' \\
\| & & F_{P,Q} & & \| \\
FX' & & \xrightarrow[F(P\otimes Q)]{} & & FZ'
\end{array}
$$

   are each valid equalities.

3. Finally, the laxity comparison cells satisfy the coherence conditions that

$$
\begin{array}{ccccc}
FX & \xrightarrow{y_{FX}} & FX & \xrightarrow{FM} & FY \\
\| & F_X & \| & 1 & \| \\
FX & \xrightarrow[Fy_X]{} & FX & \xrightarrow[FM]{} & FY \\
\| & & F_{y_X,M} & & \| \\
FX & & \xrightarrow[F(y_X\otimes M)]{} & & FX \\
\| & & \cong & & \| \\
FX & & \xrightarrow[FM]{} & & FX
\end{array}
\quad = \quad
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}\otimes FM} & FX \\
\| & & \| \\
\| & \cong & \| \\
\| & & \| \\
FX & \xrightarrow[FM]{} & FX
\end{array}
$$

holds and similarly for each $F_{M,y_X}$ and for any three composable proarrows the equality

$$
\begin{array}{ccccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ & \xrightarrow{FP} & FW \\
\| & 1 & \| & & F_{N,P} & & \| \\
FX & \xrightarrow[FM]{} & FY & & \xrightarrow[F(N\otimes P)]{} & & FW \\
\| & & & F_{M,N\otimes P} & & & \| \\
FX & & & \xrightarrow[F(M\otimes(N\otimes P))]{} & & & FW \\
\| & & & F(\mathfrak{a}_{M,N,P}) & & & \| \\
FX & & & \xrightarrow[F((M\otimes N)\otimes P)]{} & & & FW
\end{array}
\quad = \quad
\begin{array}{ccccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ & \xrightarrow{FP} & FW \\
\| & & F_{M,N} & & \| & 1 & \| \\
FX & & \xrightarrow[F(M\otimes N)]{} & & FZ & \xrightarrow[FP]{} & FW \\
\| & & & F_{M\otimes N,P} & & & \| \\
FX & & & \xrightarrow[F((M\otimes N)\otimes P)]{} & & & FW
\end{array}
$$

holds up to the omitted associators.

A lax double functor is **pseudo** if the laxity comparison cells $F_X$ and $F_{M,N}$ are invertible. It is **strict** if the laxity comparison cells are strict equalities. □

In general it is conventional that the unmodified phrase *double functor* means one that is pseudo. Lax functors are also thought of as difficult, confusing, exotic, or out-of-the-way. However, as the following examples attest, lax functors abound and for this reason they are arguably the primitive notion as opposed to strict or even pseudo.

**Example 2.2.** A lax functor $1 \to \mathbb{S}\mathbf{pan}$ on the terminal double category is the same as a small category. □

**Example 2.3.** Representable double functors $\mathbb{D}^{op} \to \mathbb{S}\mathbf{pan}$ on a small double category $\mathbb{D}$ are in general lax [Par11], [Lam21]. □

**Example 2.4.** For any double category $\mathbb{D}$, there are canonical double functors $\Delta\colon \mathbb{D} \to \mathbb{D}\times\mathbb{D}$ and $!\colon \mathbb{D} \to 1$. The former is the *diagonal* double functor, "doubling" objects, arrows, proarrows and cells. The latter is the unique double functor to the terminal double category with one object, one proarrow and no non-identity arrows or cells. Each example is genuinely pseudo. □

**Definition 2.5.** A **transformation** $\tau\colon F \Rightarrow G$ between lax double functors consists of two ordinary natural transformation $\tau_0\colon F_0 \Rightarrow G_0$ and $\tau_1\colon F_1 \Rightarrow G_1$ satisfying the following conditions.

1. The equations $\operatorname{src}\tau_M = \tau_X$ and $\operatorname{tgt}\tau_M = \tau_Y$ each hold for any proarrow $M\colon X \nrightarrow Y$.
2. For each object $X$ the equality

$$
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}} & FX \\
\| & F_X & \| \\
FX & \xrightarrow{Fy_X} & FX \\
{\scriptstyle\tau_X}\downarrow & \tau_{y_X} & \downarrow{\scriptstyle\tau_X} \\
GX & \xrightarrow[Gy_X]{} & GX
\end{array}
\quad = \quad
\begin{array}{ccc}
FX & \xrightarrow{y_{FX}} & FX \\
{\scriptstyle\tau_X}\downarrow & y_{\tau_X} & \downarrow{\scriptstyle\tau_X} \\
GX & \xrightarrow{y_{GX}} & GX \\
\| & G_X & \| \\
GX & \xrightarrow[Gy_X]{} & GX
\end{array}
$$

holds.

3. And finally for any composable proarrows $M$ and $N$, the equation

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ \\
\| & & F_{M,N} & & \| \\
FX & & \xrightarrow[F(M\otimes N)]{} & & FZ \\
\downarrow & & \tau_{M\otimes N} & & \downarrow \\
GX & & \xrightarrow[G(M\otimes N)]{} & & GZ
\end{array}
\quad = \quad
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{FN} & FZ \\
\downarrow & \tau_M & \downarrow & \tau_N & \downarrow \\
GX & \xrightarrow[GM]{} & GY & \xrightarrow[GN]{} & GZ \\
\| & & G_{M,N} & & \| \\
GX & & \xrightarrow[G(M\otimes N)]{} & & GZ
\end{array}
$$

holds.

Let $\mathbf{Dbl}_l$ denote the 2-category of double categories, lax double functors, and transformations. Let **Dbl** denote the 2-category of double categories, pseudo double functors and transformations. □

**Example 2.6.** A transformation between lax functors $1 \to \mathbb{S}\mathbf{pan}$ is a functor between small categories. □

Now that we have 2-categories of double categories, we can make the following definition, due to [Ale18, §4.2] generalizing those for bicategories from [CW87], [CKWW08].

**Definition 2.7.** A double category $\mathbb{D}$ is **precartesian** if the double functors $!\colon \mathbb{D} \to 1$ and $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ each have right adjoints in $\mathbf{Dbl}_l$. A double category $\mathbb{D}$ is **cartesian** if the double functors $!\colon \mathbb{D} \to 1$ and $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ have right adjoints in **Dbl**. □

It is worth spelling out in some detail what this means as it will help with the proofs in the last section.

# 3 Monoids and Modules in Detail

Our ultimate goal is to show that monoids and modules in a suitably structured double category are the objects and proarrows of a cartesian equipment. First we should be precise about what such monoids and modules are. Most of the following can be found in [Shu08, §11] although those definitions aren't spelled out this explicitly. Monoids and modules in double and virtual double categories go back at least to Leinster's book [Lei04] and figure prominently in more recent double-theoretic studies such as [CS10], [Sch15]. Throughout let $\mathbb{D}$ denote a double category. It doesn't hurt to think of $\mathbb{D}$ as an equipment but that won't be needed yet.

**Definition 3.1.** A **monoid** in $\mathbb{D}$ consists of an endo-proarrow $A\colon X \nrightarrow X$ together with multiplication and unit cells

$$\begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \| & & \mu & & \| \\ X & & \underset{A}{\nrightarrow} & & X \end{array} \qquad\qquad \begin{array}{ccc} X & \overset{y_X}{\longrightarrow} & X \\ \| & \epsilon & \| \\ X & \underset{A}{\nrightarrow} & X \end{array}$$

satisfying the conditions expressed by the diagrams

$$\begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \| & & \mu & & \| & 1 & \| \\ X & & \underset{A}{\nrightarrow} & & X & \underset{A}{\nrightarrow} & X \\ \| & & & \mu & & & \| \\ X & & & \underset{A}{\nrightarrow} & & & X \end{array} \quad = \quad \begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \| & 1 & \| & & \mu & & \| \\ X & \underset{A}{\nrightarrow} & X & & \underset{A}{\nrightarrow} & & X \\ \| & & & \mu & & & \| \\ X & & & \underset{A}{\nrightarrow} & & & X \end{array}$$

and

$$\begin{array}{ccccc} X & \overset{y_A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \downarrow & \epsilon & \downarrow & 1 & \downarrow \\ X & \underset{A}{\nrightarrow} & X & \underset{A}{\nrightarrow} & X \\ \| & & \mu & & \| \\ X & & \underset{A}{\nrightarrow} & & X \end{array} \quad = \quad \begin{array}{ccc} X & \overset{A}{\nrightarrow} & X \\ \| & & \| \\ \| & 1 & \| \\ \| & & \| \\ X & \underset{A}{\nrightarrow} & X \end{array} \quad = \quad \begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{y_A}{\nrightarrow} & X \\ \| & 1 & \Downarrow & \epsilon & \| \\ X & \underset{A}{\nrightarrow} & X & \underset{A}{\nrightarrow} & X \\ \| & & \mu & & \| \\ X & & \underset{A}{\nrightarrow} & & X \end{array}$$

ignoring some of the associativity and unitor isos coming with the structure of $\mathbb{D}$. Display a monoid as a quadruple $(X, A, \mu, \epsilon)$. If no ambiguity arises, reference such a monoid by the name of its proarrow, namely, in this case $A$. □

**Definition 3.2.** A **homomorphism** of monoids, displayed $(X, A, \mu, \epsilon) \to (Y, B, \nu, \eta)$ consists of a morphism $f\colon X \to Y$ and a cell

$$\begin{array}{ccc} X & \overset{A}{\nrightarrow} & X \\ {\scriptstyle f}\downarrow & \phi & \downarrow{\scriptstyle f} \\ Y & \underset{B}{\nrightarrow} & Y \end{array}$$

satisfying the conditions

$$\begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ f\downarrow & \phi & f\downarrow & \phi & \downarrow f \\ Y & \underset{B}{\nrightarrow} & Y & \underset{B}{\nrightarrow} & Y \\ \| & & \nu & & \| \\ Y & & \overset{B}{\longrightarrow} & & Y \end{array} \qquad = \qquad \begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \| & & \mu & & \| \\ X & & \overset{A}{\nrightarrow} & & X \\ f\downarrow & & \phi & & \downarrow f \\ Y & & \underset{B}{\nrightarrow} & & Y \end{array}$$

and

$$\begin{array}{ccc} X & \overset{y_A}{\nrightarrow} & X \\ \| & \epsilon & \| \\ X & \overset{A}{\nrightarrow} & X \\ f\downarrow & \phi & \downarrow f \\ Y & \underset{B}{\nrightarrow} & Y \end{array} \qquad = \qquad \begin{array}{ccc} X & \overset{y_A}{\nrightarrow} & X \\ f\downarrow & y_f & \downarrow f \\ Y & \overset{y_B}{\nrightarrow} & Y \\ \| & \eta & \| \\ Y & \underset{B}{\nrightarrow} & Y \end{array}$$

Display such a homomorphism as a pair $(f, \phi)$. Let **Mon**$(\mathbb{D})$ denote the category of monoids and homomorphisms in $\mathbb{D}$ with the evident identity morphisms and compositions. □

**Example 3.3.** A homomorphism in $\mathbb{S}$**pan** is a functor of small categories. □

**Example 3.4.** A homomorphism between monoids in $\mathbb{A}$**b** is a unital ring homomorphism. □

**Definition 3.5.** A **bimodule** between monoids in $\mathbb{D}$, displayed as $M\colon A \nrightarrow B$, consists of an underlying proarrow $M\colon X \nrightarrow Y$ and left and right action cells

$$\begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{M}{\nrightarrow} & Y \\ \| & & \lambda & & \| \\ X & & \underset{M}{\nrightarrow} & & Y \end{array} \qquad\qquad \begin{array}{ccccc} X & \overset{M}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y \\ \| & & \rho & & \| \\ X & & \underset{M}{\nrightarrow} & & Y \end{array}$$

satisfying the conditions expressed by the diagrams

$$\begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X & \overset{M}{\nrightarrow} & Y \\ \| & 1 & \| & & \lambda & & \| \\ X & \underset{A}{\nrightarrow} & X & & \underset{M}{\nrightarrow} & & Y \\ \| & & & \lambda & & & \| \\ X & & & \underset{M}{\nrightarrow} & & & Y \end{array} \qquad = \qquad \begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X & \overset{M}{\nrightarrow} & Y \\ \| & & \mu & & \| & 1 & \| \\ X & & \underset{A}{\nrightarrow} & & X & \underset{M}{\nrightarrow} & Y \\ \| & & & \lambda & & & \| \\ X & & & \underset{M}{\nrightarrow} & & & Y \end{array}$$

and

$$\begin{array}{ccccccc} X & \overset{M}{\nrightarrow} & X & \overset{B}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y \\ \| & & \rho & & \| & 1 & \| \\ X & & \underset{M}{\nrightarrow} & & Y & \underset{B}{\nrightarrow} & Y \\ \| & & & \rho & & & \| \\ X & & & \underset{M}{\nrightarrow} & & & Y \end{array} \qquad = \qquad \begin{array}{ccccccc} X & \overset{M}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y \\ \| & 1 & \| & & \nu & & \| \\ X & \underset{M}{\nrightarrow} & Y & & \underset{B}{\nrightarrow} & & Y \\ \| & & & \rho & & & \| \\ X & & & \underset{B}{\nrightarrow} & & & Y \end{array}$$

and finally

$$\begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{M}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y \\ \| & & \lambda & & \| & 1 & \| \\ X & & \underset{M}{\nrightarrow} & & Y & \underset{B}{\nrightarrow} & Y \\ \| & & & \rho & & & \| \\ X & & & \underset{M}{\nrightarrow} & & & Y \end{array} \qquad = \qquad \begin{array}{ccccccc} X & \overset{A}{\nrightarrow} & X & \overset{M}{\nrightarrow} & Y & \overset{B}{\nrightarrow} & Y \\ \| & 1 & \| & & \rho & & \| \\ X & \underset{A}{\nrightarrow} & X & & \underset{M}{\nrightarrow} & & Y \\ \| & & & \lambda & & & \| \\ X & & & \underset{M}{\nrightarrow} & & & Y \end{array}$$

expressing that the actions are compatible. Indicate such a module by a quintuple $(X, Y, M, \lambda, \rho)$. But if no confusion will arise, denote it by the underlying proarrow $M\colon X \nrightarrow Y$. ☐

**Example 3.6.** A bimodule in $\mathbb{A}\mathbf{b}$ is a bimodule between ordinary unital rings in which case these axioms take on the familiar form

$$a(a'm) = (aa')m \qquad (mb)b' = m(bb') \qquad (am)b = a(mb) \tag{3.1}$$

for all $a, a' \in A$, all $b, b' \in B$ and all $m \in M$. ☐

**Example 3.7.** A bimodule in $\mathbb{S}\mathbf{pan}$ is a profunctor between small categories. ☐

**Definition 3.8.** Let $A$, $B$, $C$, and $D$ denote monoids in $\mathbb{D}$ as in Definition 3.1. A **modulation** between bimodules $M\colon A \nrightarrow B$ and $N\colon C \nrightarrow D$ consists of an underlying cell

$$\begin{array}{ccc} X & \xrightarrow{M} & Y \\ {\scriptstyle f}\big\downarrow & \theta & \big\downarrow{\scriptstyle g} \\ Z & \xrightarrow[N]{} & Q \end{array}$$

where $f$ and $g$ are monoid homomorphisms as in Definition 3.2 satisfying the following two conditions.

1. Left action compatibility:

$$\begin{array}{ccccc} X & \xrightarrow{A} & X & \xrightarrow{M} & Y \\ {\scriptstyle f}\big\downarrow & \phi & \big\downarrow{\scriptstyle f} & \theta & \big\downarrow{\scriptstyle g} \\ Z & \xrightarrow[C]{} & Z & \xrightarrow[N]{} & W \\ \big\| & & \lambda & & \big\| \\ Z & & \xrightarrow[N]{} & & W \end{array} \quad = \quad \begin{array}{ccccc} X & \xrightarrow{A} & X & \xrightarrow{M} & Y \\ \big\| & & \lambda & & \big\| \\ X & & \xrightarrow[M]{} & & Y \\ {\scriptstyle f}\big\downarrow & & \theta & & \big\downarrow{\scriptstyle g} \\ Z & & \xrightarrow[N]{} & & W \end{array}$$

2. Right action compatibility:

$$\begin{array}{ccccc} X & \xrightarrow{M} & Y & \xrightarrow{B} & Y \\ {\scriptstyle f}\big\downarrow & \theta & \big\downarrow{\scriptstyle g} & \rho & \big\downarrow{\scriptstyle g} \\ Z & \xrightarrow[N]{} & W & \xrightarrow[D]{} & W \\ \big\| & & \rho & & \big\| \\ Z & & \xrightarrow[N]{} & & W \end{array} \quad = \quad \begin{array}{ccccc} X & \xrightarrow{M} & Y & \xrightarrow{B} & Y \\ \big\| & & \rho & & \big\| \\ X & & \xrightarrow[M]{} & & Y \\ {\scriptstyle f}\big\downarrow & & \theta & & \big\downarrow{\scriptstyle g} \\ Z & & \xrightarrow[N]{} & & W \end{array}$$

Let $\mathbf{Mon}(\mathbb{D})$ denote the category of bimodules and modulations in $\mathbb{D}$ with the evident identity modulations and compositions. ☐

Modulations appear a bit exotic, but there are in fact familiar examples.

**Example 3.9.** A modulation in $\mathbb{A}\mathbf{b}$ is a $A$-$B$-linear map as described in [Par21, Example 1.1]. ☐

**Example 3.10.** A modulation in $\mathbb{S}\mathbf{pan}$ is an ordinary transformation of profunctors. ☐

# 4 Plan of Attack

Aleiferi's thesis [Ale18] together with Shulman's paper [Shu08] gives us convenient sufficient conditions for verifying that $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a cartesian equipment under fairly natural conditions on $\mathbb{D}$ itself. For this we need a few preliminary definitions. In particular, we need to see that $\mathbb{M}\mathbf{od}(\mathbb{D})$ is in fact a double category.

**Definition 4.1.** A double category $\mathbb{D}$ has **local coequalizers** if each hom-category $\mathbb{D}(A, B)$ has coequalizers and these are preserved by external composition in each argument.

The main result in this connection is the following.

**Proposition 4.2.** *If $\mathbb{D}$ is an equipment with local coequalizers, then $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a double category and an equipment.*

*Proof.* □

**Definition 4.3.** A double category $\mathbb{D}$ has **local products** if each hom-category $\mathbb{D}(A, B)$ has finite products. When these exist, denote binary products using the wedge or conjunction symbol $m \wedge n$.

The first result is one giving sufficient conditions under which a double category $\mathbb{D}$ is precartesian. Complete proofs are given by those of Propositions 3.4.13, 3.4.16 and 4.1.2 in [Ale18].

**Proposition 4.4.** *Suppose that*

- *$\mathbb{D}$ is an equipment*
- *$\mathbb{D}_0$ has finite products*
- *$\mathbb{D}$ has finite products locally.*

*The category $\mathbb{D}_1$ then has finite products. Additionally, these products define lax functors $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I\colon 1 \to \mathbb{D}$. In other words, under these conditions $\mathbb{D}$ is precartesian.*

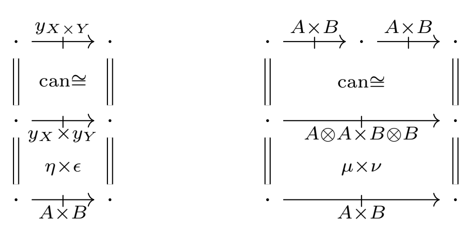
*Proof.* □

**Corollary 4.5.** *Suppose that $\mathbb{D}$ satisfies the hypotheses of Proposition 4.4. If the induced lax functors $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I\colon 1 \to \mathbb{D}$ are both pseudo double functors, then $\mathbb{D}$ is cartesian.*

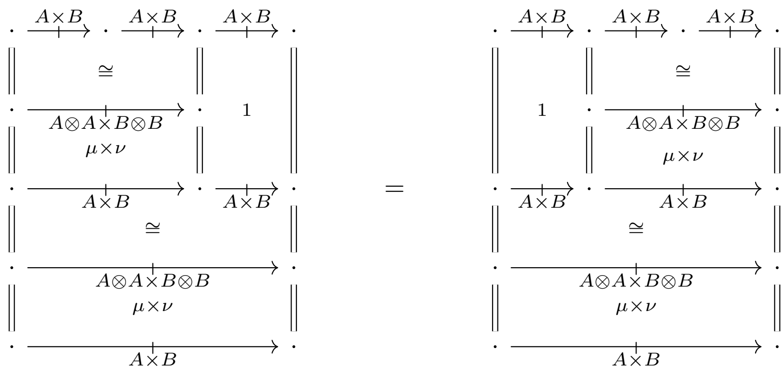
# 5 Main Results

Throughout let $\mathbb{D}$ denote a cartesian equipment with local coequalizers. In this case, we know at least that $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a double category and an equipment by Proposition 4.4. We endeavor in several steps to show that it is cartesian using the sufficient conditions of Corollary 4.5. Specifically, we need to show that $\mathbb{M}\mathbf{od}(\mathbb{D})_0$ has finite products, that $\mathbb{M}\mathbf{od}(\mathbb{D})$ has finite products locally, and that the induced lax functors are in fact pseudo. Many of the computations in the required proofs become quite repetitive, so when it is appropriate we shall show only representative calculations and leave some of the remaining verifications to the conscientious reader.

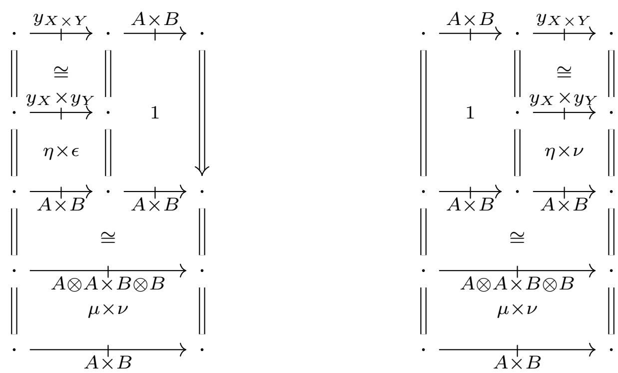
First we will show that $\mathbf{Mon}(\mathbb{D})$ has finite products. Start with a bit of discussion and set-up before the actual proof. Proposition 4.4 implies in particular that $\mathbb{D}_1$ has finite products. So, if we have two monoids, $(X, A, \mu, \eta)$ and $(Y, B, \nu, \epsilon)$, our claim is that this product $A \times N\colon X \times Y \nrightarrow X \times Y$ is the underlying object of the binary product in $\mathbf{Mon}(\mathbb{D})$. The required unit and multiplication will be the following composites:




The isos here are the canonical comparisons coming with the (pseudo!) double functors $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I\colon 1 \to \mathbb{D}$. The proof below will verify the action condition for $\mu \times \nu$ in Definition 3.1 for a monoid. This takes the form of the proposed equation



where the isos are the canonical laxity cells. These again are invertible since $\mathbb{D}$ is cartesian. Likewise, the unit condition takes the form of the statement that each of the two composites

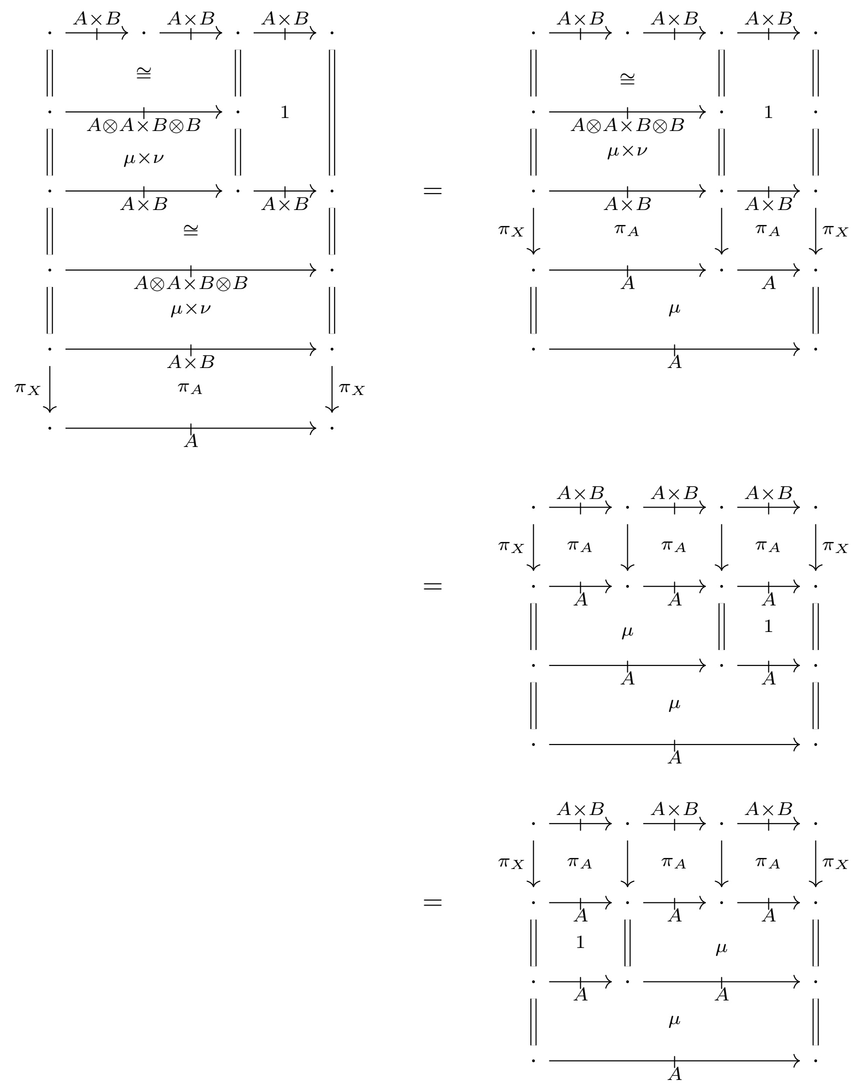


is equal to the identity on $A \times B$ composed with the appropriate unitor isomorphism in the given $\mathbb{D}$-structure. Both statements are part of the proof of the following result. Universality will be handled separately afterward.

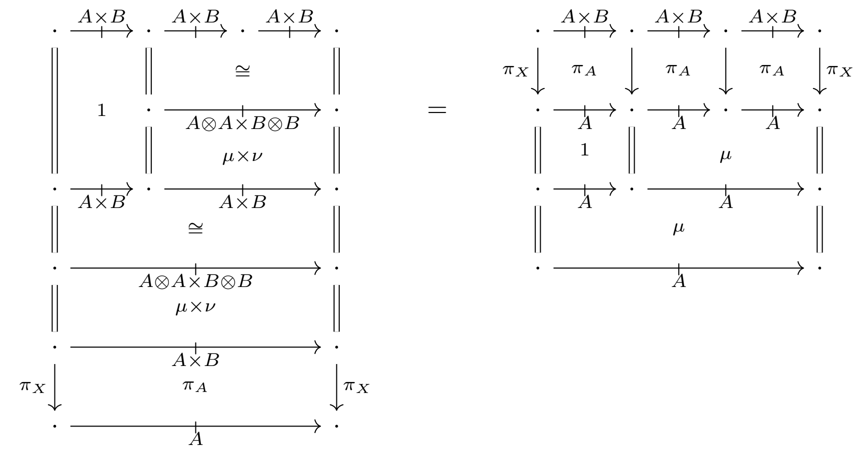
**Lemma 5.1.** *The product in $\mathbb{D}_1$ of the underlying proarrows of two monoids is again a monoid.*

*Proof.* Use the notation established in the discussion above. We will prove that the action axiom holds for $\mu \times \nu$. The strategy is to show that the projections to $A$ and to $B$ each coequalize the cells on either side;

so, by uniqueness, these cells must then be equal. First note that we have the equalities

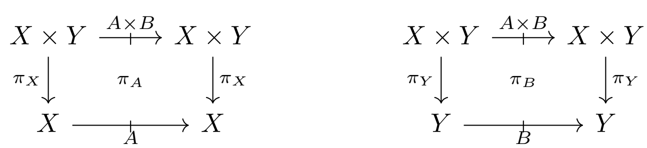


The last equality is the easy one which is justified by the fact that $\mu$ itself is a monoid operation. The first two equalities are just uses of the projection morphisms $\pi_A$ on the product morphisms $\mu \times \nu$ and the canonical isos. For example, the first equality uses first the fact that $\pi_A(\mu \times v) = \mu\pi_{A\otimes A}$ and then the fact that $\pi_{A\otimes A}$ following the canonical iso is precisely $\pi_A \otimes \pi_A$. This trick with the projections is repeated for the second equality. Now, on the other hand, by analogous reasoning, we have that

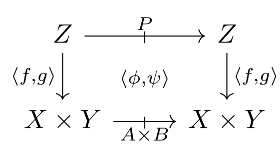


holds too. Therefore $\pi_A$ coequalizes the cells on each side of the desired equation. By a perfectly symmetric argument, $\pi_B$ does as well. Therefore, by uniqueness of the universal map to $A \times B$ in $\mathbb{D}_1$ the required equation must hold and $\mu \times \nu$ is an action in the required sense. That the proposed unit satisfies the required law is a similar argument proceeding from use of the projections and uniqueness. □

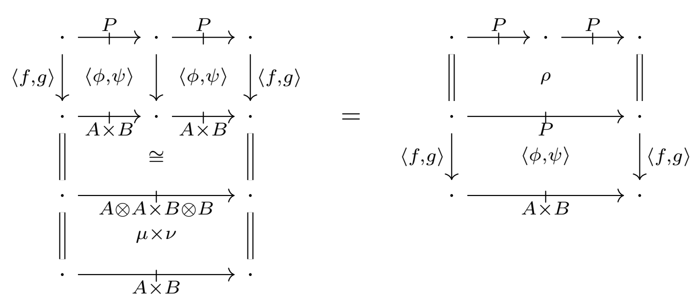
The next result is mostly just checking the universality of the previous construction in Lemma 5.1. For the most part this is a follow-your-nose type of argument on the pattern of the last proof. We shall show some representative computations. First, what are the projections and canonically induced product morphisms? Given two monoids $(X, A, \mu, \eta)$ and $(Y, B, \nu, \epsilon)$, the product $A \times B\colon X \times Y \nrightarrow X \times Y$ is again a monoid, so our claim is that the given projections



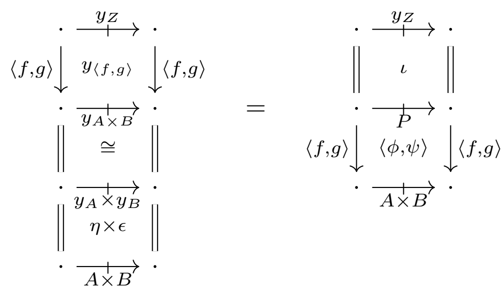
are monoid homomorphisms and that for any further monoid, say, $(Z, P, \rho, \iota)$ with homomorphisms $\phi\colon P \to A$ and $\psi\colon P \to B$, the canonically induced product morphism



given by the binary product structure in $\mathbb{D}$ is again a monoid homomorphism. The argument for the projections are almost immediate and will be handled in the proof below. Instantiating the homomorphism conditions in Definition 3.2 for $\langle \phi, \psi \rangle$ we need to show that



and

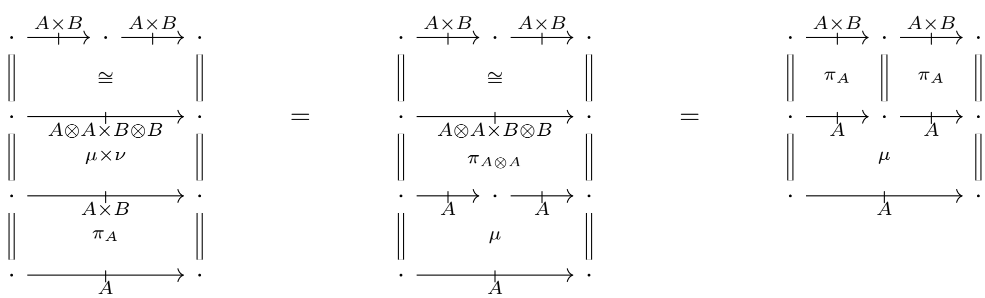



both hold. The strategy is the same as before, namely, in the proof below we shall show that the composite cells on each side of the equations above are coequalized by the projections. By uniqueness, the two sides in each case must be equal.

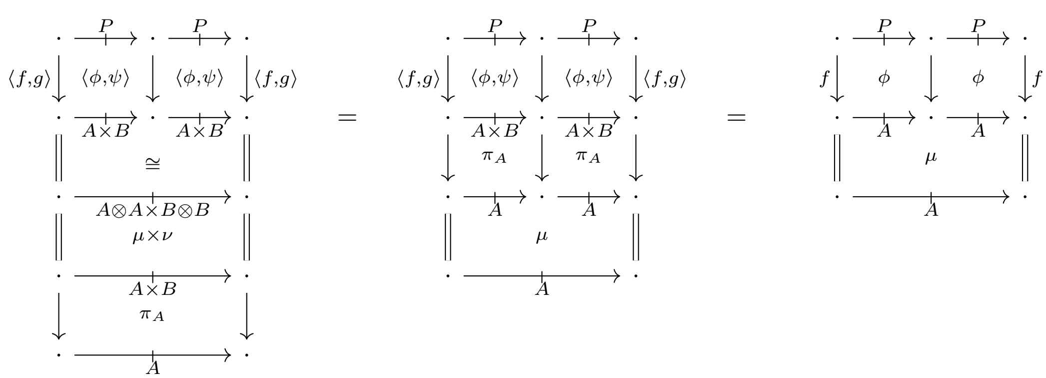
**Lemma 5.2.** **Mon**($\mathbb{D}$) *has finite products.*

*Proof.* There are two parts to the proof: first, showing the existence of a terminal object; and second showing that binary products exist. But the first is immediate As in the discussion above, since the product of the underlying proarrows of two monoids is again a monoid by Lemma 5.1, it suffices to show the universality of the construction. For this we shall show that the projections and universal map supplied by the product structure of $\mathbb{D}$ are monoid homomorphisms.

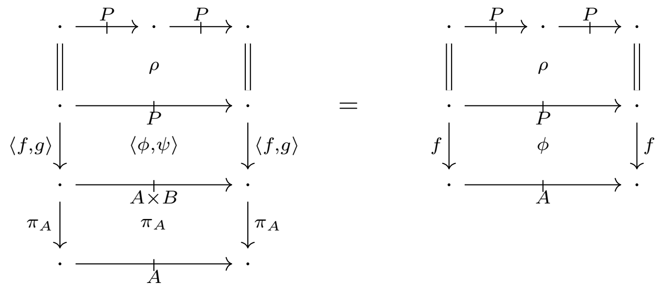
For the projections, it suffices to check that say $\pi_A$ is a homomorphism. But at least for multiplication preservation, the argument is just instantiating the *projection trick* as discussed previously and used in the prior proof:




We will leave unit-preservation to the reader. The arguments are the same for the other projection. The claim now is that $\langle \phi, \psi\rangle$ is a monoid homomorphism. We shall verify multiplication preservation as in the equation in the display above. On the one hand, we have



using the projection trick twice for the left-most equality. And on the other hand, we have that




just using the action of the projections. But the far right-hand sides of each of the last two displays are equal since $\phi$ is a monoid homomorphism. Therefore, $\pi_A$ coequalizes the cells on either side of the required equation for preservation of multiplication. By an analogous computation $\pi_B$ does as well. Consequently, by uniqueness, the required equation holds. Preservation of identities follows by a similar style of argument. □

The next goal is to show that $\mathbb{M}\mathbf{od}(\mathbb{D})$ has finite products locally.

**Lemma 5.3.** $\mathbb{M}\mathbf{od}(\mathbb{D})$ *has finite products locally.*

*Proof.* □

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